$$\text{rank}\left[\sum_{i=1}^{n}\hat{h}_i(t)\hat{b}_i, \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\hat{h}_i(t)\hat{b}_i, \ldots, \sum_{i=1}^{n}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\hat{h}_i\hat{b}_i\right]$$

If $\hat{h}_i (i=1,\ldots,n)$ are linearly independent, it is easy to show that

$$\text{rank}\left[\sum_{i=1}^{n}\hat{h}_i(t)\hat{b}_i, \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\hat{h}_i(t)\hat{b}_i, \ldots, \sum_{i=1}^{n}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\hat{h}_i\hat{b}_i\right]$$

$$=\text{rank}[\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n] = n$$

This completes the proof.

In duality, a theorem of observability for the reducible system can be established as follows:

Theorem 4: System [Eq. (8)] is totally observable only if $(CF_\beta J^\alpha)^T$ $(\alpha=0,\ldots,n-1, \beta=1,\ldots,r)$ has rank n and if its corresponding functions $\hat{h}_i(t)$ $(i=1,\ldots,n)$ are linearly independent.

The interpretation of this result is that a large number of time-variant system modes can be controlled by a few actuators and sensors. In fact, the number of actuators (or sensors) is determined by the number of columns of the influence matrix B (or rows of the matrix C); however, it is still required that control devices be located so as to satisfy the rank conditions and function independence tests of the theorems. Some trial and error of locations may be necessary subject to physical constraints, as well as the economic concern.

Discussion and Conclusion

A large majority of large flexible dynamic systems are nonlinear over a wide range of amplitudes of the dynamical quantities involved. Even though these systems contain nonlinearities in their normal range of performance, a linearized form of their state equation will be a valid approximation, provided that the state variables involved do not vary too widely from their nominal states about which linearization takes place. Moreover, the results obtained from such a linear model should give a qualitative idea of the way in which system performance will differ from model behavior. For these reasons, we introduce in this paper a linear model for a large flexible structure.

If the linear system is reducible in the sense of Lyapunov, the time-variant characteristic matrix Γ can be transformed to a time-invariant Jordan matrix. Our new results show that controllability and observability of a large flexible spacecraft can be determined from that simple Jordan form and the properties of the Lyapunov transformation.

It is clear that the eigenvalues of Γ, when Γ is constant, play a very similar part in the solution of the variational equation to that played by the characteristic exponent when Γ is reducible. It is natural, therefore, to extend the use of the term "characteristic exponents" so as to include the case when the elements of Γ are constant. In a problem in which Γ is a constant matrix, the exponents are its eigenvalues. Consequently, this may initiate the idea of updating or adapting some existing control theory from the time-invariant system to the reducible dynamic system.

Acknowledgments

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Evaluation of Mass Properties by Finite Elements

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Introduction

WHEN determining the dynamic behavior of a space body a fundamental requirement is the accurate calculation of the body's mass properties. The usual method for performing this operation is to view the body as an assemblage of small elements for which standard formulas exist, evaluate the mass properties at the element level from these formulas and recover in total the properties of the body as a whole. As the shape of the body becomes more complicated the accuracy of this calculation may diminish. The method presented herein follows this same basic philosophy differing only in the calculation at the element level, wherein a finite element type analysis is employed.

The application of isoparametric finite elements for the approximation of boundary value problems defined over irregular domains is a standard procedure. Unlike this usual application for finite elements the method presented in this note does not seek the solution of a boundary-value problem but rather the evaluation of some simple integrals defined over the domain.

Analysis

Let Ω be the domain and x_i, $i=1,2,3$ be a global coordinate system with basis vectors $e_i, i=1,2,3$. The particular properties of interest are the mass M, the first mass moment Q and the moment of inertia I. When expanded against the basis e_i these properties take the form

$$M=\int_\Omega \rho \mathrm{d}x \tag{1a}$$

$$Q=Q_i e_i \qquad Q_i=\int_\Omega \rho x_i \mathrm{d}x \tag{1b}$$

$$I=I_{ij} e_i \otimes e_j \qquad I_{ij}=\int_\Omega \rho(\delta_{ij}x_k x_k - x_i x_j)\mathrm{d}x \tag{1c}$$

where ρ is the density of the body and δ_{ij} is the Kronecker delta; in addition, the usual Cartesian summation convention has been employed.

The domain Ω can be viewed as an assemblage of E finite elements $\{\Omega_e\}_{e=1}^{E}$, two such elements being shown in Fig. 1. The number of nodes on element e is denoted as N_e; for the elements shown in Fig. 1 N_e is 8 and 20, respectively. The location of an arbitrary point x in Ω_e can be interpolated with respect to the known locations of the nodes denoted as x_i^α, $\alpha=1,\ldots N_e$. This interpolation takes the form

$$x_i = \sum_{\alpha=1}^{N_e} N_\alpha(\xi) x_i^\alpha \tag{2}$$

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where $N_\alpha(\xi)$ are the interpolation basis functions (shape functions) defined with respect to the natural coordinates ξ_i, $i=1,2,3$. These natural coordinates have the property that when $x_i = x_i^\alpha$, then $\xi_i = \xi_i^\alpha$ where $-1 \le \xi_i \le 1$.

In addition $N_\alpha(\xi)$ has the property that

$$N_\alpha(\xi^\beta) = \delta_\alpha^\beta \tag{3}$$

As an example consider the eight node brick of Fig. 1a for which $N_\alpha(\xi)$ is of the form

$$N_\alpha(\xi) = \tfrac{1}{8}(1+\xi_1^\alpha\xi_1)(1+\xi_1^\alpha\,\xi_2)(1+\xi_3^\alpha\xi_3) \tag{4}$$

The interpolation in Eq. (2) can be considered as a mapping of a cube (master element) $\bar{\Omega}$ in ξ-space onto the element Ω_e in x-space. Interpolation functions for higher order elements can be found in Ref. 1 and a discussion of associated mathematics in Ref. 2. The Jacobian tensor associated with the above formulation is of the form

$$J = \frac{\partial x}{\partial \xi} = \sum_{\alpha=1}^{N_e} \frac{\partial N_\alpha(\xi)}{\partial \xi_j} x_i^\alpha \, e_i \otimes e_j \tag{5}$$

Letting J represent the determinant of $\boldsymbol{J}$, then measure in the two spaces is related by

$$dx = dx_1\,dx_2\,dx_3 = J d\xi_1\,d\xi_2\,d\xi_3 = J d\xi \tag{6}$$

In addition the same interpolation can be carried out for the density, i.e.,

$$\rho(x) = \sum_{\gamma=1}^{N_e} N_\gamma(\xi)\rho^\gamma \tag{7}$$

where ρ^γ is the prescribed density at node γ. Upon substituting Eqs. (2) (4) and (7) into Eq. (1) we obtain

$$M = \int_{\bar{\Omega}} J d\xi \tag{8}$$

$$Q_i = \sum_{\alpha=1}^{N_e}\sum_{\gamma=1}^{N_e} Q_{\alpha\gamma} x_i^\alpha \rho^\gamma \tag{9a}$$

$$\bar{Q}_{\alpha\gamma} = \int_{\bar{\Omega}} N_\alpha(\xi) N_\gamma(\xi) J d\xi \tag{9b}$$

$$I_{ij} = \sum_{\alpha=1}^{N_e}\sum_{\beta=1}^{N_e}\sum_{\gamma=1}^{N_e} [\delta_{ij} x_k^\alpha \bar{I}_{\alpha\beta\gamma} x_k^\beta - x_i^\alpha \bar{I}_{\alpha\beta\gamma} x_j^\beta]\rho^\gamma \tag{10a}$$

$$\bar{I}_{\alpha\beta\gamma} = \int_{\bar{\Omega}} N_\alpha(\xi) N_\beta(\xi) N_\gamma(\xi) J d\xi \tag{10b}$$

Hence in place of evaluating a volume integral over Ω_e, we now need evaluate the integral of a polynomial defined over the master element $\bar{\Omega}$; this integal is now in a form appropriate for the Gauss-Legendre quadrature rule. In order to determine the number of integration points required for an exact evaluation, one must look at the degree of the polynomial integrand. As an example, consider the case of the eight node brick element (Fig. 1a) with constant density; then from Eq. (4) we see that

$$N_\alpha = 0(\xi_1\xi_2\xi_3) \qquad J_{11} = 0(\xi_2\xi_3)$$

which implies that

$$J = 0(\xi_1^2\xi_2^2\xi_3^2)$$

Hence the integrand of Eq. (10b) does not contain a polynomial of degree greater than 4 in any direction. This implies that the integrals in Eqs. (8-10) are evaluated exactly with $3\times3\times3$ rule. It should be noted that the number of integration points increases dramatically with order of the element; for example, in the case of a twenty-node brick element (Fig. 1b) with variable density, a $6\times6\times6$ integration rule is required. However, when higher order elements are employed a much coarser mesh is required; in addition, some savings in quadrature can be realized by employing transition (variable node) elements. These elements would have two curved and four straight surfaces and hence would require a $6\times6\times3$ rule.

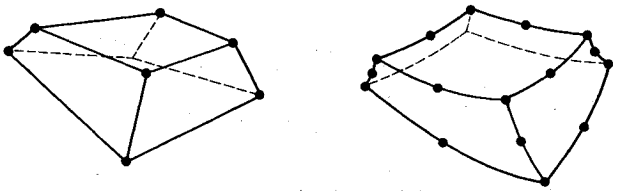

Fig. 1 a) Eight node brick element; b) twenty node solid element.

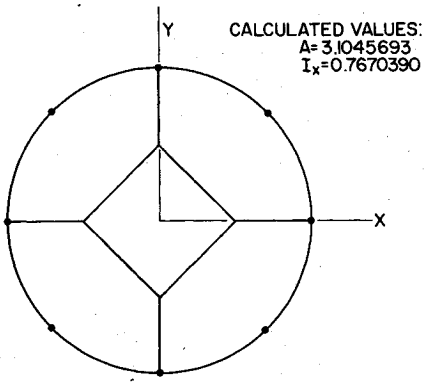

Fig. 2a) Five element coarse mesh.

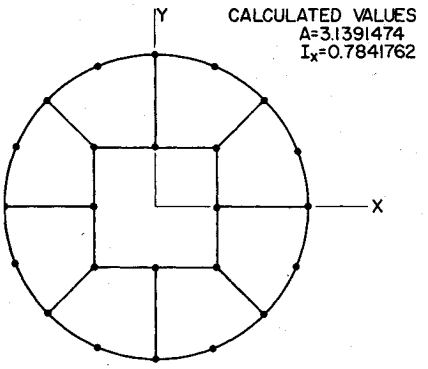

Fig. 2b) Nine element fine mesh.

For the case of a plate element, a local coordinate system on the plate surface is established via a rotation matrix and the same procedure as outlined above (except in two dimensions) is carried out to determine the plate's mass properties with respect to this local system. These can then be related back to the global coordinate system by means of the rotation matrix and the parallel axis theorem. It should be mentioned that these elements can easily be used to evaluate the area moment of a beam's cross section for use in the flexure equation.

Computationally the evaluation of the mass properties follows identically with the formulation of a structural member's stiffness matrix; i.e., an outer loop on the integration points, the evaluation of the Jacobian, followed by an inner loop on the coordinates. As the mass properties for each element are determined, they are summed so that when the calculation for the last element is complete the properties for the body as a whole have been determined. Post processing involves simply determining the centroid and the

principal axis for the body. It should be mentioned that since we are neither assembling nor solving a system of equations, the large run times and massive storage requiremens normally associated with finite element analysis are not present in this case. Hence, a mass properties code of the form outlined above can easily be implemented on a minicomputer.

To illustrate the accuracy of the procedure, consider the simple problem of employing plate elements to determine the area properties for a circular section of unit radius. Two such meshes are shown in Figs. 2a and 2b. For the coarser five-element mesh (Fig. 2a) the error in area and moment of inertia are, respectively, 1.178% and 2.338%. While for the finer nine-element mesh (Fig. 2b), the corresponding errors are 0.078% for the area and 0.156% for the moment of inertia. For either solution less than 1 s of CPU time was required on a UNIVAC 1108.

Conclusion

Presented in this Note is a procedure for the accurate evaluation of mass properties for a space body. The procedure is particularly effective when the geometry of the body is irregular. The methodology employed is a perturbation to the standard implementation of iso-parametric finite elements.

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Improved Method for Solving the Algebraic Riccati Equation

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Introduction

SINCE linear quadratic regulator theory came into vogue in the early 1960's, many authors have suggested methods to solve the resulting algebraic Riccati equation. One of the methods that has been most successful is the eigenvector decomposition method first proposed by MacFarlane[1] and by Potter.[2] In this method, the eigenvalues and eigenvectors of the Euler-Lagrange system are determined. The eigenvectors associated with eigenvalues whose real parts are all of the same sign are partitioned into two matrices. These matrices form a set of linear equations which yield the Riccati equation solution.

The success of the method hinges on the requirement that the partitioned eigenvector matrices be nonsingular. In the case when one or more of the eigenvalues is repeated, the resulting matrices may be singular. The singularity can be removed by using the generalized eigenvectors. However, this method is not entirely satisfactory. In the case when the eigenvalues are nearly equal, the partitioned eigenvector matrices are not singular, but they remain ill conditioned. This ill conditioning can lead to errors in the computed solution. Also, small perturbations in the system matrix elements can lead to drastic changes in the partitioned eigenvector matrices, which in turn causes poor numerical stability.

In order to alleviate these difficulties, the following method is proposed:

1) Determine the coefficient matrix H of the Euler-Lagrange system. For the algebraic Riccati equation,

$$SA+A^TS+C^TC-SBB^TS=0$$

the matrix H is given by

$$H=\begin{bmatrix} A, & -BB^T \\ -C^TC, & -A^T \end{bmatrix}$$

2) Calculate the eigenvalues of the coefficient matrix using the highly stable QR algorithm.[3,4]

3) Again use the QR algorithm with shifts of origin beginning at the previously computed eigenvalues with positive real parts. This procedure will compose the system into the form

$$H=P^T\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}P \qquad (1)$$

where

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is orthogonal and U_{22} has eigenvalues with positive real parts.

4) The symmetric non-negative solution, S, of the Riccati equation is found by solving the linear equation (see Theorem 1 in Appendix)

$$P_{11}S=P_{12} \qquad (2)$$

The matrices P_{11} and P_{12} are nonsingular when the system is controllable and observable (see Theorem 2 in Appendix).

If the system has unobservable or uncontrollable modes, then varying the control will not affect the performance index through this part of the system. Thus, these modes can be ignored and the design can proceed using only the completely observable and controllable part of the system.

The major advantage of the method lies in the inherent orthogonality of the P matrix. The QR algorithm produces an extremely stable decomposition into the form of Eq. (1). The P matrix is also not overly sensitive to small perturbations in the system matrix elements or to small perturbations in the computed eigenvalues.

A second advantage of the method is that less computation is required to determine the P matrix than is required to determine the matrix of eigenvectors.

Results

To exemplify the method, consider the following simple optimal control problem:

$$\min_u J=\int_0^\infty\left(x^T\begin{bmatrix} 0 & 0 \\ 0 & q^2 \end{bmatrix}x+u^2\right)dt$$

subject to

$$\dot{x}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}x+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad (4)$$

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